QUASI-EINSTEIN METRICS ON HYPERSURFACE FAMILIES

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Abstract. We construct quasi-Einstein metrics on some hypersurface families. The hypersurfaces are circle bundles over the product of Fano, Kähler-Einstein manifolds. The quasi-Einstein metrics are related to various gradient Kähler-Ricci solitons constructed by Dancer and Wang and some Hermitian, non-Kähler, Einstein metrics constructed by Wang and Wang on the same manifolds.

1. Introduction

1.1. **Motivation and definitions.** This article[1] is concerned with a generalisation of Einstein metrics that in some sense interpolates between Einstein metrics and Ricci solitons, namely, quasi-Einstein metrics.

Definition 1.1. Let M^n be a smooth manifold and g be a complete Riemannian metric. The metric g is called quasi-Einstein if it solves

$$Ric(g) + Hess(u) - \frac{1}{m} du \otimes du + \frac{\epsilon}{2} g = 0, \tag{1.1}$$

where $u \in C^{\infty}(M)$, $m \in (1, \infty]$ and ϵ is a constant.

It is clear that if u is constant then we recover the notion of an Einstein metric; we will refer to these metrics as trivial quasi-Einstein metrics. By letting the constant m go to infinity we can also recover the definition of a gradient Ricci soliton. In line with the terminology used for Ricci solitons, we will refer to the quasi-Einstein metrics with $\epsilon < 0$, $\epsilon = 0$ and $\epsilon > 0$ as shrinking, steady and expanding respectively.

There has been a great deal of effort invested in finding non-trivial examples of Ricci solitons on compact manifolds. However, they remain rare and the only known examples are Kähler. Due to work the work of Hamilton [13] and Perelman [19], non-trivial Ricci solitons on compact manifolds must be shrinking gradient Ricci solitons. The first non-trivial examples were constructed independently by Koiso and Cao on $\mathbb{CP}^1$-bundles over complex projective spaces [3, 15]. These examples were subsequently generalised by Chave and Valet [7] and Pedersen, Tønneson-Freidman and Valent [18] who found Kähler-Ricci solitons on the projectivisation of various line bundles over a Fano Kähler-Einstein base. The reader should note that what we call a Ricci soliton is referred to as a quasi-Einstein metrics in the papers [7] and [18]. Recently Dancer and Wang generalised these examples by constructing some Kähler Ricci solitons on various hypersurface families where the hypersurface is a circle bundle over the product of Fano Kahler-Einstein manifolds [9]. The solitons found by Dancer and Wang were also independently constructed by Apostolev, Calderbank, Gauduchon and Tønneson-Freidman [1].

In the complete non-compact case Feldman, Ilmanen and Knopf [11] found shrinking gradient Kähler-Ricci solitons on certain line bundles over $\mathbb{CP}^n$. Steady gradient Kähler-Ricci solitons were first constructed on $\mathbb{C}^n$ by Cao [3] (the $n = 1$ case was first found by Hamilton [12]). Cao also found steady gradient Kähler-Ricci solitons on the blow up of $\mathbb{C}^n/\mathbb{Z}_n$ at the origin. Expanding gradient Kähler-Ricci solitons have been found by Cao on $\mathbb{C}^n$ [4] and by Feldman, Ilmanen and Knopf on the blow ups of $\mathbb{C}^n/\mathbb{Z}_k$ for $k = n+1, n+2, \ldots,$ [11]. Examples were also found by Pedersen, Tønneson-Freidman and Valent on the total space of holomorphic line bundles over Kahler-Einstein manifolds with negative scalar curvature [18]. As in the compact case, these examples have been generalised by Dancer and Wang who constructed shrinking, steady and expanding Kähler-Ricci solitons on various vector bundles over the product of Kähler-Einstein manifolds [9].

In the recent work [6] Case suggested that there should be quasi-Einstein analogues of Dancer-Wang's solitons. He points out that the quasi-Einstein analogue of Koiso-Cao, Chave-Valent and Pedersen-Tønneson-Freidman-Valent type solitons was already constructed by Lü, Page and Pope [16]. The purpose of this article is to show that Dancer-Wang's solitons indeed have quasi-Einstein analogues. However it is better to think of these metrics as quasi-Einstein analogues of various Hermitian, non-Kähler, Einstein metrics constructed by Wang and Wang on these spaces [20]. The Wang-Wang Einstein metrics generalise a construction originating with Page [17] and Berard-Bergery [2]. We now state the precise results we wish to prove. Non-trivial steady or expanding quasi-Einstein metrics can only occur on non-compact manifolds [14]. In the non-compact case we have the following which is the quasi-Einstein analogue of theorem 1.6 in [20]:

Theorem 1.2. *Let $(V_i, J_i, h_i), 1 \leq i \leq r, r \geq 3$, be Fano Kähler-Einstein manifolds with complex dimension n_i and first Chern class $p_i a_i$ where $p_i > 0$ and a_i are indivisible classes in $H^2(V_i, \mathbb{Z})$. Let V_1 be a complex projective space with normalised Fubini-Study metric i.e. $p_1 = (n_1 + 1)$. Let P_q denote the principal $\mathbb{S}^1$-bundle over $V_1 \times \ldots \times V_r$ with Euler class $\pm\pi_1^*(a_1) + \sum_{i=2}^{i=r} q_i \pi_i^*(a_i)$, i.e. $q_1^2 = 1$.*

(1) Suppose $(n_1 + 1)|q_i| < p_i$ for $2 \leq i \leq r$ then, for all $m > 1$, there exists a non-trivial, complete, steady, quasi-Einstein metric on the

total space of the $\mathbb{C}^{n_1+1}$-bundle over $V_2 \times ... \times V_r$ corresponding to P_q.

(2) *For all $m > 1$ there exists at least one one-parameter family of non-trivial, complete, expanding, quasi-Einstein metrics on the total space of the $\mathbb{C}^{n_1+1}$-bundle over $V_2 \times ... \times V_r$ corresponding to P_q.*

For the compact case we have the following analogue of theorem 1.2 in [20].

Theorem 1.3. *Let $(V_i, J_i, h_i), 1 \leq i \leq r, r \geq 3$, be Fano Kähler-Einstein manifolds with complex dimension n_i and first Chern class $p_i a_i$ where $p_i > 0$ and a_i are indivisible classes in $H^2(V_i, \mathbb{Z})$. Let V_1 and V_r be a complex projective space with normalised Fubini-Study metrics. Let P_q denote the principal $\mathbb{S}^1$-bundle over $V_1 \times ... \times V_r$ with Euler class $\pm\pi_1^*(a_1) + \sum_{i=2}^{i=r-1} q_i \pi_i^*(a_i) \pm \pi^*(a_r)$, i.e. $|q_1| = |q_r| = 1$.*

Suppose that $|q_i|(n_1 + 1) < p_i$ and $|q_i|(n_r + 1) < p_i$ for $2 \leq i \leq r - 1$ and that there exists $\chi = (\chi_1, \chi_2, ..., \chi_r)$ where $|\chi_i| = 1$, $\chi_1 = -\chi_r = 1$ such that

$$\int_{-(n_1+1)}^{(n_r+1)} \left(\chi_1 x + \frac{p_1}{|q_1|}\right)^{n_1} \left(\chi_2 x + \frac{p_2}{|q_2|}\right)^{n_2} ... \left(\chi_r x + \frac{p_r}{|q_r|}\right)^{n_r} x dx < 0, \tag{1.2}$$

then, for all $m > 1$ there exists a non-trivial, shrinking quasi-Einstein metric on M_q, the space obtained from $P_q \times_{S_1} \mathbb{CP}^1$ by blowing-down one end to $V_2 \times ... \times V_r$ and the other end to $V_1 \times ... \times V_{r-1}$.

We remark that the Futaki invariant (evaluated on the holomorphic vector field $f(t)\partial_t$ in the notation of the next section) is given by

$$\int_{-(n_1+1)}^{(n_r+1)} \left(\frac{p_1}{q_1} - x\right)^{n_1} \left(\frac{p_2}{q_2} - x\right)^{n_2} ... \left(\frac{p_r}{q_r} - x\right)^{n_r} x dx.$$

If this integral vanishes then Dancer-Wang construct a Kähler-Einstein metric on M_q.

Finally we note that none of the metrics we find are Kähler. Indeed there is a rigidity result due to Case-Shu-Wei [5] that says, on compact manifolds, Kähler-quasi-Einstein metrics are trivial i.e. Kähler-Einstein.

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2. Proof of main theorems

2.1. **Derivation of equations.** We use the same notation as above. We consider the manifold

$M_0 = (0, l) \times P_q$. Let θ be the principal $U(1)$-connection on P_q with curvature $\Omega = \sum_{i=1}^{r} q_i \pi^* \eta_i$ where η_i is the Kähler form of the metric h_i. We form the 1-parameter family of metrics on P_q

$$g_t = f^2(t)\theta \otimes \theta + \sum_{i=1}^{i=r} g_i^2(t)\pi^* h_i$$

and we then form the metric $\bar{g} = dt^2 + g_t$ on M_0. The group $U(1)$ acts on M_0 by isometries and generates a Killing field Z. We define a complex structure J on M_0 by $J(\partial_t) = -f^{-1}(t)Z$ on the vertical space of θ and by lifting the complex structure from the base on the horizontal spaces of θ.

Lemma 2.1. *Let M_0 be as above and let $v = e^{-\frac{u}{m}}$. Then the quasi-Einstein equations in this setting are given by:*

$$\frac{\ddot{f}}{f} + \sum_{i=1}^{i=r} 2n_i \frac{\ddot{g}_i}{g_i} + m\frac{\ddot{v}}{v} = \frac{\epsilon}{2}, \tag{2.1}$$

$$\frac{\ddot{f}}{f} + \sum_{i=1}^{i=r} \left(2n_i \frac{\dot{f}\dot{g}_i}{fg_i} - \frac{n_i q_i^2}{2} \frac{f^2}{g_i^4} \right) + m\frac{\dot{f}\dot{v}}{fv} = \frac{\epsilon}{2}, \tag{2.2}$$

$$\frac{\ddot{g}_i}{g_i} - \left(\frac{\dot{g}_i}{g_i}\right)^2 + \frac{\dot{f}\dot{g}_i}{fg_i} + \sum_{j=1}^{j=r} 2n_j \frac{\dot{g}_i\dot{g}_j}{g_i g_j} - \frac{p_i}{g_i^2} + \frac{q_i^2 f^2}{2g_i^4} + m\frac{\dot{g}_i\dot{v}}{g_i v} = \frac{\epsilon}{2}. \tag{2.3}$$

In order that (M, g, u) be a quasi-Einstein manifold, as well as equation (1.1), u must also satisfy an integrability condition that essentially comes from the second Bianchi identity (c.f. Lemma 3.4 in [9]). The form we use here is given in Case [6] using the Bakry-Émery Laplacian:

$$\Delta_u := \Delta - \langle \nabla u, \cdot \rangle.$$

Lemma 2.2 (Kim-Kim [14] Corollary 3)**.** *Let (M, g, u) be a quasi-Einstein manifold then there exists a constant μ such that*

$$\Delta_u \left(\frac{u}{m}\right) + \frac{\epsilon}{2} = -\mu e^{\frac{2u}{m}}. \tag{2.4}$$

In the notation above (recalling $v = e^{-\frac{u}{m}}$) this condition becomes

$$\mu = v\ddot{v} + v\dot{v}\left(\frac{\dot{f}}{f} + \sum_i 2n_i \frac{\dot{g}_i}{g_i}\right) + (m-1)\dot{v}^2 - \frac{\epsilon}{2}v^2. \tag{2.5}$$

The constant μ enters into the discussion of Einstein warped products when m is an integer. If (M, g, u) is a quasi-Einstein manifold with constant μ coming from (2.4) and (F^m, h) is an Einstein manifold with constant μ, then $(M \times F^m, g \oplus v^2 h)$ is an Einstein metric with constant $-\epsilon/2$ as in equation (1.1) (c.f. [14]).

Introducing the moment map change of variables as in [9] and [20] yields the following set of equations:

Proposition 2.3. *Let s be the coordinate on $I = (0, l)$ such that $ds = f(t)dt$, $\alpha(s) = f^2(t)$, $\beta_i(s) = g_i^2(t)$, $\phi(s) = v(t)$ and $V = \prod_{i=1}^{i=r} g_i^{2n_i}(t)$. Then the equations (2.1),(2.2),(2.3) and (2.5) transform to the following:*

$$\frac{1}{2}\alpha'' + \frac{1}{2}\alpha'(\log V)' + \alpha \sum_{i=1}^{r} n_i \left(\frac{\beta_i''}{\beta_i} - \frac{1}{2}\left(\frac{\beta_i'}{\beta_i}\right)^2 \right) + m\left(\frac{\alpha\phi''}{\phi} + \frac{\alpha'\phi'}{2\phi}\right) = \frac{\epsilon}{2}, \tag{2.6}$$

$$\frac{1}{2}\alpha'' + \frac{1}{2}\alpha'(\log V)' - \alpha \sum_{i=1}^{i=r} \frac{n_i q_i^2}{2\beta_i^2} + m\frac{\alpha'\phi'}{2\phi} = \frac{\epsilon}{2}, \tag{2.7}$$

$$\frac{1}{2}\frac{\alpha'\beta_i'}{\beta_i} + \frac{1}{2}\alpha\left(\frac{\beta_i''}{\beta_i} - \left(\frac{\beta_i'}{\beta_i}\right)^2\right) + \frac{1}{2}\frac{\alpha\beta_i'}{\beta_i}(\log V)' - \frac{p_i}{\beta_i} + \frac{q_i^2\alpha}{2\beta_i^2} + m\frac{\alpha}{2}\frac{\beta_i'\phi'}{\beta_i\phi} = \frac{\epsilon}{2}, \tag{2.8}$$

$$\phi\left(\phi''\alpha + \frac{\phi'\alpha'}{2}\right) + \phi\phi'\left(\frac{\alpha'}{2} + (\log V)'\alpha\right) + (m-1)(\phi')^2\alpha - \frac{\epsilon}{2}\phi^2 = \mu. \tag{2.9}$$

Equating (2.6) and (2.7) we obtain

$$-m\frac{\phi''}{\phi} = \sum_{i=1}^{i=r} n_i \left(\frac{\beta_i''}{\beta_i} - \frac{1}{2}\left(\frac{\beta_i'}{\beta_i}\right)^2 + \frac{q_i^2}{2\beta_i^2} \right) \tag{2.10}$$

Following [9, 20] we look for solutions that satisfy

$$\frac{\beta_i''}{\beta_i} - \frac{1}{2}\left(\frac{\beta_i'}{\beta_i}\right)^2 + \frac{1}{2}\frac{q_i^2}{\beta_i^2} = 0.$$

This condition can be geometrically interpreted as saying that the curvature of $\overline{g}$ is J-invariant in the sense that $\overline{Rm}(J\cdot, J\cdot, J\cdot, J\cdot) = \overline{Rm}(\cdot,\cdot,\cdot,\cdot)$ where J is the complex structure on M_0. Imposing this forces ϕ to be a linear function of s. We write $\phi(s) = \kappa_1(s + \kappa_0)$ for some constants $\kappa_0, \kappa_1 \in \mathbb{R}$. Hence (2.9) becomes

$$\alpha' + \alpha((\log V)' + \frac{(m-1)}{(s+\kappa_0)}) = \frac{\epsilon(s+\kappa_0)}{2} + \frac{\mu}{\kappa_1^2(s+\kappa_0)}. \tag{2.11}$$

Accordingly there are two classes of solution β_i:

$$\beta_i = A_i(s+s_0)^2 - \frac{q_i^2}{4A_i}$$

or

$$\beta_i = \pm q_i(s+\sigma_i)$$

where $A_i \neq 0$ and σ_i are constants. We note that the case $\beta_i = -q_i(s+\sigma_i)$ corresponds to the metric $\bar{g}$ being Kähler with respect to the complex structure. Hence the rigidity result of Case-Shu-Wei rules out having any solutions of this form (in fact choosing β_i of this form leads to inconsistency).

If we input $\beta_i = A_i(s+s_0)^2 - \frac{q_i^2}{4A_i}$ into (2.8) we see that

$$\alpha' + \alpha\left((\log V)' + m(\log\phi)' - \frac{1}{(s+s_0)}\right) = \frac{\epsilon}{2}(s+\kappa_0) + \frac{E^*}{(s+\kappa_0)}$$

where

$$E^* := \frac{8A_ip_i - \epsilon q_i^2}{8A_i^2}.$$

Comparing with equation (2.11) we see that solutions are consistent providing $\kappa_0 = s_0$ and

$$\frac{\mu}{\kappa_1^2} = E^* = \frac{8A_ip_i - \epsilon q_i^2}{8A_i^2}.$$

Solving gives

$$\alpha(s) = V^{-1}(s+\kappa_0)^{1-m}\int_0^s V(s+\kappa_0)^{m-2}\left(E^* + \frac{\epsilon}{2}(s+\kappa_0)^2\right)ds. \quad (2.12)$$

2.2. **Compactifying M_0.** We recall that $V_1 = \mathbb{CP}^{n_1}$ and we are adding in the manifold $V_2\times ... \times V_r$ at the point $s=0$. We refer the reader to the discussion immediately after equation (4.17) in [9]. In a nutshell, in order for the metric to extend smoothly we require that

$$\alpha(0) = 0, \alpha'(0) = 2, \beta_1(0) = 0 \text{ and } \beta_1'(0) = 1.$$

As we are using $\beta_1(s) = A_1(s+\kappa_0)^2 - \frac{q_1^2}{4A_1}$ we must have $A_1 = \frac{1}{2\kappa_0}$ and $|q_1| = 1$. We also have normalised so that $p_1 = n_1 + 1$ hence the consistency conditions become

$$E^* = \frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2}(4(n_1+1) - \epsilon\kappa_0) = \frac{8A_ip_i - \epsilon q_i^2}{8A_i^2} \text{ for } 2 \le i \le r.$$

2.3. **Steady quasi-Einstein metrics.** In this case $\epsilon = 0$. Setting $V_1 = \mathbb{CP}^{n_r}$ and compactifying we obtain a $\mathbb{C}^{n_1+1}$-vector bundle over $V_2\times ... \times V_r$. In order that $\beta_i(0) > 0$ on $I = [0,\infty)$ we must have $A_i > 0$ and

$$E^* = \frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2}(4(n_1+1)) = \frac{p_i}{A_i} \text{ for } 2 \le i \le r.$$

Hence $A_i = \frac{p_i}{E^*}$ and

$$\beta_i(s) = \frac{p_i}{E^*}(s+\kappa_0)^2 - \frac{E^* q_i^2}{4p_i}.$$

It is clear that in order for $\beta_i(0) > 0$ we must have

$$(n_1+1)|q_i| < p_i \text{ for } 2 \le i \le r.$$

In order to ensure the metrics are complete it is sufficient to check that the integral

$$t = \int_0^s \frac{dx}{\sqrt{\alpha(x)}} \quad (2.13)$$

diverges as $s \to \infty$ (this says that geodesics cannot reach the boundary at infinity and have finite length). If we compute the function $\alpha(s)$ we see that it is asymptotic to a positive constant K. Hence the above integral diverges and we obtain a complete quasi-Einstein metric for all $m > 1$ generalising the non-Kähler, Ricci-flat ones constructed in [20]. Choosing a different value of E^* simply varies the metric by homothety.

2.4. **Expanding quasi-Einstein metrics.** Here we take $\epsilon = 1$ to factor out homothety. Again the manifolds in question are $\mathbb{C}^{n_1+1}$-vector bundles over $V_2 \times ... \times V_r$. Here the consistency conditions become

$$E^* = \frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2}(4(n_1+1) - \kappa_0) = \frac{8A_i p_i - q_i^2}{8A_i^2} \text{ for } 2 \leq i \leq r.$$

If $|q_i|(n_1+1) \leq p_i$ then we choose $0 < E^* < 2(n_1+1)^2$,

$$\kappa_0 = 2(n_1+1) + 2\sqrt{(n_1+1)^2 - \frac{E^*}{2}}$$

and

$$A_i = \frac{1}{2E^*}\left(p_i + \sqrt{p_i^2 - \frac{E^* q_i^2}{2}}\right).$$

In order that $\beta_i(0) > 0$ we require $2\kappa_0 A_i > |q_i|$ for $2 \leq i \leq r$. This can be seen as

$$2\left(2(n_1+1) + 2\sqrt{(n_1+1)^2 - \frac{E^*}{2}}\right)\frac{1}{2E^*}\left(p_i + \sqrt{p_i^2 - \frac{E^* q_i^2}{2}}\right) > \frac{2(n_1+1)p_i}{E^*} > |q_i|.$$

In the case that $|q_i|(n_1+1) < p_i$ we note also that

$$\left(1 + \sqrt{1 - \frac{E^* q_i^2}{2p_i^2}}\right) > \left(1 + \sqrt{1 - \frac{E^*}{2(n_1+1)^2}}\right),$$

hence,

$$2\left(2(n_1+1) - 2\sqrt{(n_1+1)^2 - \frac{E^*}{2}}\right)\frac{1}{2E^*}\left(p_i + \sqrt{p_i^2 - \frac{E^* q_i^2}{2}}\right) >$$

$$\frac{4p_i(n_1+1)}{2E^*}\left(1 - \sqrt{1 - \frac{E^*}{2(n_1+1)^2}}\right)\left(1 + \sqrt{1 + \frac{E^*}{2(n_1+1)^2}}\right) = \frac{p_i}{(n_1+1)} > |q_i|.$$

Therefore if we have the strict inequality $|q_i|(n_1+1) < p_i$ then we can also choose

$$\kappa_0 = 2(n_1+1) - 2\sqrt{(n_1+1)^2 - \frac{E^*}{2}}.$$

If $|q_i|(n_1+1) > p_i$ then we can choose $0 < E^* < 2(n_1+1)^2 \min(p_2^2/q_2^2, ..., p_r^2/q_r^2)$. If we also choose

$$\kappa_0 = 2(n_1+1) + 2\sqrt{(n_1+1)^2 - \frac{E^*}{2}}$$

and

$$A_i = \frac{1}{2E^*} p_i + \sqrt{p_i^2 - \frac{E^* q_i^2}{2}},$$

then $\beta_i(0) > 0$. We can also choose $E^* < 0$ in this case. Completeness follows as $\alpha(s)$ is asymptotic to Ks^2 for a positive constant K and so the integral (2.13) diverges. Hence we find complete, quasi-Einstein analogues of the non-Kahler, Einstein metrics constructed in [20].

2.5. **Shrinking quasi-Einstein metrics.** In order to factor out homothety we take $\epsilon = -1$ and so the consistency conditions are

$$\frac{\mu}{\kappa_1^2} = \frac{\kappa_0}{2}(4(n_1+1)+\kappa_0) = \frac{8A_i p_i + q_i^2}{8A_i^2} \text{ for } 2 \leq i \leq r.$$

We split the discussion into the compact case and the non-compact, complete case. For the compact case we consider I to be the finite interval $[0, s_*]$. We set $V_r = \mathbb{CP}^{n_r}$ and at the point $s = s_*$ we add in the manifold $V_1 \times ... \times V_{r-1}$. For the metric to extend smoothly we require that $q_r = 1, p_r = n_r + 1$ and $-1 = 2A_r(s_* + \kappa_0)$. Putting these into the consistency conditions we see that

$$\kappa_0(4(n_1+1)+\kappa_0) = (s_* + \kappa_0)^2 - 4(n_r+1)(s_* + \kappa_0)$$

and hence

$$s_* = \sqrt{\kappa_0(4(n_1+1)+\kappa_0) + 4(n_r+1)^2} - \kappa_0 + 2(n_r+1).$$

We note that if $n_1 = n_r$ then $s_* = 4(n_1+1)$. For the time being we note that $s_* = s_*(E^*)$ and β_i is completely determined by E^* once we have chosen the value of q_i^2 and the sign of A_i. The value A_i is given by

$$A_i = \frac{1}{2E^*}\left(p_i + \chi_i \sqrt{p_i^2 + \frac{E^* q_i^2}{2}}\right)$$

where $\chi_i = 1$ if $A_i > 0$ and $\chi_i = -1$ if $A_i < 0$. In order to have a quasi-Einstein metric we must be able choose a value of $E^* > 0$ such that the integral

$$\int_0^{s_*(E^*)} \prod_{i=0}^{i=r} \left[\left((s+\kappa_0)^2 - \frac{q_i^2}{4A_i^2}\right)^{n_i}\right] (s+\kappa_0)^{m-2} \left(E^* - \frac{1}{2}(s+\kappa_0)^2\right) ds = 0.$$

Changing coordinates to

$$x = \frac{1}{2}(s+\kappa_0) - ((n_1+1)^2 + \frac{E^*}{2})^{1/2},$$

then the above integral becomes (ignoring constants)

$$F(E^*) = \int_{-(n_1+1)}^{x_*(E^*)} \prod_{i=0}^{i=r} P_i(x)(x+((n_1+1)^2+\frac{E^*}{2})^{1/2})^{m-2}(x^2+2x((n_1+1)^2+\frac{E^*}{2})^{1/2})+(n_1+1)^2)ds$$

where

$$P_i(x) = \left(x^2 + 2x((n_1+1)^2 + \frac{E^*}{2})^{1/2} + (n_1+1)^2 + \frac{2p_i(\chi_i\sqrt{p_i^2 + \frac{E^* q_i^2}{2}} - p_i)}{q_i^2} \right)^{n_i}$$

and

$$x_*(E^*) = (n_r+1) + (\frac{E^*}{2} + (n_r+1)^2)^{1/2} - (\frac{E^*}{2} + (n_1+1)^2)^{1/2}.$$

We will compute the limit $\lim_{E^* \downarrow 0} F(E^*)$ and the limit $\lim_{E^* \to \infty} F(E^*)$. We begin with 0. We note that as $m > 1$ the function $f(x) = (x+(n_1+1)^{m-2}$ is integrable on $[-(n_1+1), x(E^*)]$ so by the dominated convergence theorem we can evaluate the integral of the limit. This is given by

$$S \int_{-(n_1+1)}^{2(n_r+1)-(n_1+1)} \prod_{\chi_i=-1} [x+(n_1+1)]^{2n_i} \prod_{\chi_j=1} \left[\frac{4p_i^2}{q_i^2} - (x+(n_1+1))^2 \right]^{n_j} (x+(n_1+1))^m dx,$$

where

$$S = (-1)^{\sum_{\chi_i=-1} n_i}.$$

The hypothesis on the p_i and q_i mean that the sign of $\lim_{E^* \downarrow 0} F(E^*)$ is that of S.

For $E^* \to \infty$ we consider

$$\lim_{E^* \to \infty} F(E^*)(E^*)^{\frac{1}{2}(1-m-\sum_{\chi_i=-1} n_i)} = K(-1)^{\sum_{\chi_i=-1} n_i} \int_{-(n_1+1)}^{(n_r+1)} \prod_{i=1}^{i=r} \left[\chi_i x + \frac{p_i}{|q_i|} \right]^{n_i} x dx,$$

where K is a positive constant. Hence if we can choose χ_i so that

$$\int_{-(n_1+1)}^{(n_r+1)} \prod_{i=1}^{i=r} \left[\chi_i x + \frac{p_i}{|q_i|} \right]^{n_i} x dx < 0,$$

we can find an $E^* > 0$ such that $\alpha(s_*) = 0$. A discussion similar to that in [9] and [20] shows that this is enough to ensure we have smooth quasi-Einstein metrics.

3. Examples and future work

We end with an example of theorem 1.3, some discussion of the geometry of the quasi-Einstein metrics constructed and a discussion of possible sources future compact examples.

3.1. **An example.** We consider an example that is also considered in [9]. They consider a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^2 \times \mathbb{CP}^2$. In theorem 1.3 this corresponds to taking $r = 4, n_1 = n_4 = 0, n_2 = n_3 = 2$ and $p_2 = p_3 = 3$. Hence to apply the theorem we must consider $|q_2|, |q_3| < 3$. They take $(q_2, q_3) = (1, -2)$. The Futaki invariant is given by

$$\int_{-1}^{1} (3-x)^2(\frac{3}{2}+x)^2 x dx$$

which they calculate is 7.8. This means that

$$\int_{-1}^{1}(3+x)^2(\frac{3}{2}-x)^2 x dx = -7.8 < 0$$

and we have non-trivial quasi-Einstein metrics on this space for all $m > 1$.

3.2. **Remarks on the geometry of the quasi-Einstein metrics.** In [9] section 4, the authors comment on the geometry at infinity of their examples of steady and expanding gradient Kähler-Ricci solitons. In particular they conclude that their steady examples are asympotically parabolic and that the expanding examples are asymptotically conical. We recall that the examples of steady quasi-Einstein metrics constructed in theorem 1.2 have $\alpha(s) \sim K$ for some positive constant K and so the following asymptotic behaviour holds (ignoring multiplicative constants)

$$f(t) = O(1) \text{ and } g_i(t) \sim t.$$

In the expanding case we recall that $\alpha(s) \sim Ks^2$ and so we have

$$f(t) \sim e^t \text{ and } g_i(t) \sim e^t.$$

3.3. **Future families.** The space $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ fits into the framework of theorem 1.3 as a non-trivial $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. On this space there is the Page metric, the Koiso-Cao soliton and the quasi-Einstein metrics of theorem 3 (originally due to Lü-Page-Pope). The space $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$ also admits a non-Kähler, Einstein metric due to Chen, LeBrun and Weber [8] and a Kähler-Ricci soliton due to Wang and Zhu [21]. It would seem reasonable that there should be a family of quasi-Einstein analogues to these metrics. The metrics on $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$ are not cohomogeneity-one but do have an isometric action by $\mathbb{T}^2$. One observation is that the Lü-Page-Pope quasi-Einstein metrics are conformally Kähler (as any $U(2)$-invariant metric on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ is). The Chen-LeBrun-Weber metric is also conformally Kähler (a fact orginally proved by Derdzinski [10]) and so one might hope that the same would be true for analogues of the Lü-Page-Pope metrics on $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$. Both the Page and Chen-LeBrun-Weber metrics are conformal to extremal Kähler metrics which satisfy a PDE that 'occurs naturally' in Kähler geometry. It would be an interesting first step to try and find an analogous PDE/ODE for the Kähler metrics that are conformal to the Lü-Page-Pope metrics. The author hopes to take up the existence questions in a future work.

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