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QUATERNIONIC GEOMETRY IN DIMENSION EIGHT

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ABSTRACT. We describe the 8-dimensional Wolf spaces as cohomogeneity one $SU(3)$ -manifolds, and discover perturbations of the quaternion-kähler metric on the simply-connected 8-manifold $G_2/SO(4)$ that carry a closed fundamental 4-form but are not Einstein.

To Nigel Hitchin on the occasion of his 70th birthday

1. INTRODUCTION

Of the “fundamental geometries” captured by Berger’s list of holonomy groups, the quaternionic unitary group stands out in that Riemannian manifolds with holonomy in $Sp(n)Sp(1)$ are Einstein but not Ricci-flat, unless locally hyperkähler. Excluding the latter case, the study of these quaternion-kähler manifolds splits into two cases, depending on the sign of the scalar curvature. The negative case is fairly flexible [1, 2, 9, 10], but the situation of positive scalar curvature is extremely rigid. In fact, it is conjectured that a complete positive quaternion-kähler manifold is necessarily one of the symmetric spaces that were first described by Wolf [23]. This rigidity suggests a quest for ways of weakening the holonomy condition. It turns out that dimension 8 harbours a particularly natural type of almost quaternion-kähler manifold.

An appealing way of expressing an almost quaternionic Hermitian structure is to say that our 8-manifold admits a 4-form that is pointwise linearly equivalent to

$$\Omega = \frac{1}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2), \quad (1.1)$$

where $(\omega_1, \omega_2, \omega_3)$ is the standard hyperkähler triplet on $\mathbb{R}^8 \cong \mathbb{H}^2$:

$$\begin{cases} \omega_1 = dx^{12} + dx^{34} + dx^{56} + dx^{78}, \\ \omega_2 = dx^{13} + dx^{42} + dx^{57} + dx^{86}, \\ \omega_3 = dx^{14} + dx^{23} + dx^{58} + dx^{67}. \end{cases}$$

In these terms, the quaternion-kähler condition then amounts to Ω being parallel for the Levi-Civita connection, $\nabla\Omega = 0$.

Swann [22] observed that it is possible to have Ω non-parallel and closed (and so harmonic), but that closedness of the fundamental 4-form implies quaternion-kähler in dimension at least 12. Using exterior differential systems, Bryant [5] analysed the local existence of the “harmonic” $Sp(2)Sp(1)$ -structures and showed that solutions exist in abundance, even though the PDE system at first looks overdetermined. In fact, involutivity of the exterior differential system can be deduced by observing that the contraction of Ω with any vector $v \in \mathbb{R}^8$ induces a stable form on the

quotient $\mathbb{R}^8/\langle v \rangle$, cf. [7, 8]. In [21] the third author provided the first compact non-parallel example of such a geometry. Later many more examples have followed [14, 8] by reducing the internal symmetry group from the quaternionic unitary group to its intersection with $\mathrm{SO}(6)$ and $\mathrm{SO}(7)$.

These previously known examples of non-parallel harmonic $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -manifolds all have infinite fundamental group and associated metric of negative scalar curvature. A natural question is whether such structures with positive scalar curvature exist on simply-connected manifolds.

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2. THE WOLF SPACES

In Wolf’s construction [23] one starts with a compact centreless simple Lie group G with Lie algebra \mathfrak{g} and choice of Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. One then picks a maximal root $\beta \in \mathfrak{t}$ and considers an associated $\mathfrak{sp}(1)$ and its centraliser \mathfrak{l}_1 in \mathfrak{g} . The Lie algebra $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{l}_1$ will model the holonomy algebra of a symmetric space: if we let G be the simply-connected compact simple Lie group corresponding to \mathfrak{g} and K the compact subgroup generated by \mathfrak{k} , then G/K is a compact symmetric quaternion-kähler manifold, a so-called *Wolf space*, with holonomy K . The associated quaternionic structure on the tangent space is generated by the subgroup $\mathrm{Sp}(1) \subset K$.

2.1. Quaternionic projective plane. The model space for a quaternion-kähler 8-manifold is the quaternionic projective plane

$$\mathbb{H}\mathrm{P}(2) = \frac{\mathrm{Sp}(3)}{\mathrm{Sp}(2) \times \mathrm{Sp}(1)}.$$

In Wolf’s terms, we can describe this as follows.

Choose the Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{sp}(3)_{\mathbb{C}}$ spanned by the three elements

$$H_j = E_{i,i} - E_{i+3,i+3},$$

where the matrix $E_{k,\ell}$ has only non-zero entry, equal to 1, at position (k, ℓ) . Then let $L_j \in \mathfrak{h}^*$ be the element satisfying $L_j(H_i) = \delta_{ji}$. The corresponding roots of $\mathfrak{sp}(3)_{\mathbb{C}}$ are the vectors $\pm L_i \pm L_j$. The associated eigenspaces are spanned by

$$\begin{aligned} X_{i,j} &= E_{i,j} - E_{3+j,3+i}, \\ Y_{i,j} &= E_{i,3+j} + E_{j,3+i}, \quad Z_{i,j} = E_{3+i,j} + E_{3+j,i}, \\ U_i &= E_{i,3+i}, \quad V_i = E_{3+i,i} \end{aligned}$$

where $i \neq j$ in the first two rows.

A real structure σ is determined by

$$\begin{aligned}\sigma(H_i) &= -H_i, \sigma(X_{i,j}) = -X_{j,i}, \\ \sigma(Y_{i,j}) &= -Z_{i,j}, \sigma(Z_{i,j}) = -Y_{i,j}, \\ \sigma(U_i) &= -V_i, \sigma(V_i) = -U_i,\end{aligned}$$

and we can therefore choose a basis of $\mathfrak{sp}(3)$ given by

$$\left\{ \underbrace{iH_k}_{A_k}, \underbrace{X_{k,\ell} - X_{\ell,k}}_{P_{k+\ell-2}}, \underbrace{i(X_{k,\ell} + X_{\ell,k})}_{P_{k+\ell+1}}, \underbrace{Y_{k,\ell} - Z_{k,\ell}}_{Q_{k+\ell-2}}, \underbrace{i(Y_{k,\ell} + Z_{k,\ell})}_{Q_{k+\ell+1}}, \underbrace{U_k - V_k}_{R_k}, \underbrace{i(U_k + V_k)}_{R_{k+3}} \right\}.$$

In these terms Wolf's highest root $\mathfrak{sp}(1)$ is given by

$$\mathfrak{sp}(1) = \langle A_1, R_1, R_4 \rangle,$$

and its centraliser is the copy of $\mathfrak{sp}(2)$ described as

$$\mathfrak{k}_1 = \langle A_2, A_3, P_3, P_6, Q_3, Q_6, R_2, R_3, R_5, R_6 \rangle.$$

We now have the direct sum decomposition

$$\mathfrak{sp}(3) = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ and

$$\mathfrak{p} = \mathfrak{k}^\perp = \langle P_1, P_2, P_4, P_5, Q_1, Q_2, Q_4, Q_5 \rangle.$$

The basis

$$\frac{1}{4\sqrt{2}}P_1, \frac{1}{4\sqrt{2}}P_4, \frac{1}{4\sqrt{2}}Q_1, \frac{1}{4\sqrt{2}}Q_4, \frac{1}{4\sqrt{2}}P_2, \frac{1}{4\sqrt{2}}P_5, \frac{1}{4\sqrt{2}}Q_2, \frac{1}{4\sqrt{2}}Q_5,$$

is orthonormal for the Killing metric on the subspace $\mathfrak{p} = (\mathfrak{sp}(1) \oplus \mathfrak{sp}(2))^\perp$, and determines an adapted frame for the $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structure. Its dual basis is given by

$$\begin{aligned}f^1 &= 4\sqrt{2}p_1, f^2 = 4\sqrt{2}p_4, f^3 = 4\sqrt{2}q_1, f^4 = 4\sqrt{2}q_4, f^5 = 4\sqrt{2}p_2, \\ f^6 &= 4\sqrt{2}p_5, f^7 = 4\sqrt{2}q_2, f^8 = 4\sqrt{2}q_5,\end{aligned}$$

where p_1, \dots, q_5 is the dual basis of P_1, \dots, Q_5 .

2.2. Complex Grassmannian. Consider next the complex Grassmannian of planes in \mathbb{C}^4 :

$$\mathrm{Gr}_2(\mathbb{C}^4) = \frac{\mathrm{SU}(4)}{\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(2))}.$$

In order to describe Wolf's structure on this space, we begin by considering $\mathrm{SU}(4)_{\mathbb{C}} = \mathrm{SL}(4, \mathbb{C})$ with its usual basis:

$$H_i = E_{i,i} - E_{i+1,i+1}, X_1 = E_{1,2}, X_2 = E_{1,3}, \dots, X_6 = E_{3,4}, Y_1 = E_{2,1}, \dots, Y_6 = E_{4,3}.$$

From the real structure σ , given by $\sigma(H_i) = -H_i$, $\sigma(X_i) = -Y_i$, $\sigma(Y_i) = -X_i$, we see that a basis of $\mathfrak{su}(4)$ can be described as

$$\mathfrak{su}(4) = \langle \underbrace{iH_j}_{A_j}, \underbrace{X_j - Y_j}_{C_j}, \underbrace{i(X_j + Y_j)}_{B_j} \rangle.$$

In these terms, Wolf's highest root $\mathfrak{sp}(1)$ reads

$$\mathfrak{sp}(1) = \langle i(H_1 + H_2 + H_3), C_3, B_3 \rangle$$

and its centraliser is

$$\mathfrak{k}_1 = \langle i(H_1 - H_3), iH_2, C_4, B_4 \rangle.$$

As a result we have the direct sum decomposition

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k} = \mathfrak{sp}(1) \oplus \mathfrak{k}_1$ and

$$\mathfrak{p} = \mathfrak{k}^\perp = \langle C_1, B_1, C_2, B_2, C_5, B_5, C_6, B_6 \rangle.$$

We see that the orthonormal frame

$$\frac{1}{4}C_1, \frac{1}{4}B_2, \frac{1}{4}C_5, \frac{1}{4}B_5, \frac{1}{4}C_2, \frac{1}{4}B_2, \frac{1}{4}C_6, \frac{1}{4}B_6$$

is adapted to the $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structure. Letting c_1, \dots, b_6 denote the dual basis of C_1, \dots, B_6 , we then have an $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -adapted coframe f^1, \dots, f^8 given by $f^1 = 4c_1$, and so forth.

2.3. The exceptional Wolf space. We finally turn to Wolf's construction of a quaternion-kähler structure on

$$\frac{\mathrm{G}_2}{\mathrm{SO}(4)}.$$

This is slightly more involved, due to the more complicated nature of \mathfrak{g}_2 .

First we need to choose a suitable basis of $(\mathfrak{g}_2)_{\mathbb{C}}$. We shall follow [11], where the long roots are given by

$$\alpha_2 = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right) = -\beta_2, \alpha_5 = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right) = -\beta_5, \alpha_6 = (0, \sqrt{3}) = -\beta_6,$$

and the short roots are

$$\alpha_1 = (1, 0) = -\beta_1, \alpha_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = -\beta_3, \alpha_4 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = -\beta_4.$$

We shall pick $\beta = (0, \sqrt{3})$ as the highest root.

The real structure σ of $\mathfrak{g}_2^{\mathbb{C}}$ is determined by

$$\sigma(H_i) = -H_i, \quad \sigma(X_j) = -Y_j, \quad \sigma(Y_j) = -X_j,$$

and a basis of \mathfrak{g}_2 is therefore given by

$$\{A_1 = iH_1, A_2 = iH_2, W_j = X_j - Y_j, Z_j = i(X_j + Y_j): 1 \leq j \leq 6\}.$$

The highest root $\mathfrak{sp}(1)$ is given by

$$\mathfrak{sp}(1) = \langle A_1 + 2A_2, W_6, Z_6 \rangle,$$

and its centraliser in \mathfrak{g}_2 is given by

$$\mathfrak{l}_1 = \langle A_1, W_1, Z_1 \rangle.$$

We now have the direct sum decomposition

$$\mathfrak{g}_2 = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\mathfrak{p} = \mathfrak{k}^\perp = \langle W_2, W_3, W_4, W_5, Z_2, Z_3, Z_4, Z_5 \rangle.$$

In particular, the basis

$$W_2, Z_2, -W_5, -Z_5, \frac{1}{\sqrt{3}}W_3, \frac{1}{\sqrt{3}}Z_3, -\frac{1}{\sqrt{3}}W_4, -\frac{1}{\sqrt{3}}Z_4,$$

which is orthonormal for the Killing form on the subspace $\mathfrak{p} = \mathfrak{so}(4)^\perp$, determines an adapted frame for this $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structure. As in the previous cases, we let f^i denote its dual coframe, meaning $f^1 = w_2$, etc.

3. COHOMOGENEITY ONE $\mathrm{SU}(3)$ -ACTIONS

Each of the 8-dimensional Wolf spaces M admits a cohomogeneity one $\mathrm{SU}(3)$ -action, which was studied by Gambioli [12], see also [19]. We summarise and elaborate on key facts below. In each case, the action comes from the embedding $\mathrm{SU}(3) \subset G$. In order to explicitly describe the orbits of this $\mathrm{SU}(3)$ -action, we choose an element $Z \in \mathfrak{p} \cap \mathfrak{su}(3)^\perp$ and write $\gamma(t) = \exp(tZ)$. Then the $\mathrm{SU}(3)$ -orbits of $\gamma(t)$ are given by

$$\iota_t: \mathrm{SU}(3) \rightarrow M, \quad g \mapsto g\gamma(t)\mathbb{K}.$$

As the quotient map

$$\pi: G \rightarrow M, \quad g \mapsto g\mathbb{K}$$

has $\ker \pi_{*g} = L_{g*}(\mathfrak{k})$, we can identify $T_{\gamma(t)\mathbb{K}}M$ with $L_{\gamma(t)*}(\mathfrak{p})$, or simply \mathfrak{p} where left translation is then understood. It follows that we can identify ι_{t*} with the map $\mathfrak{su}(3) \rightarrow \mathfrak{p}$ given by

$$X \mapsto [\mathrm{Ad}(\gamma(t)^{-1})(X)]_{\mathfrak{p}}.$$

Since Z is orthogonal to $\mathfrak{su}(3) \subset \mathfrak{g}$, it is clear from invariance of the Killing form that $L_{\gamma(t)*}(Z)$ is orthogonal to the $\mathrm{SU}(3)$ -orbit of $\gamma(t)$ for all t , and altogether the cohomogeneity one action infinitesimally is described by the mapping

$$\mathfrak{su}(3) \oplus \mathbb{R} \rightarrow \mathfrak{p}, \quad X \mapsto [\mathrm{Ad}(\gamma(t)^{-1})(X)]_{\mathfrak{p}}, \quad \frac{\partial}{\partial t} \mapsto Z. \quad (3.1)$$

Now, given the adapted quaternion-kähler frame on $T_e\mathbb{K}M$, we can use (3.1) to pull this back to $\mathfrak{su}(3) \oplus \mathbb{R}$ and thereby get a description of the Wolf space structure that is adapted to the cohomogeneity one setting.

Before doing so, we fix some conventions for $\mathfrak{su}(3)$: in the following e^1, \dots, e^8 will always denote a basis of $\mathfrak{su}(3)^*$ such that the following structure equations hold:

$$\begin{aligned} de^1 &= -e^{23} - e^{45} + 2e^{67}, & de^2 &= e^{13} + e^{46} - e^{57} - \sqrt{3}e^{58}, \\ de^3 &= -e^{12} - e^{47} + \sqrt{3}e^{48} - e^{56}, & de^4 &= e^{15} - e^{26} + e^{37} - \sqrt{3}e^{38}, \\ de^5 &= -e^{14} + e^{27} + \sqrt{3}e^{28} + e^{36}, & de^6 &= -2e^{17} + e^{24} - e^{35}, \\ de^7 &= 2e^{16} - e^{25} - e^{34}, & de^8 &= -\sqrt{3}(e^{25} - e^{34}). \end{aligned} \quad (3.2)$$

In terms of matrices, we can express the dual basis e_1, \dots, e_8 as

$$\begin{aligned} e_1 &= E_{21} - E_{12}, \quad e_2 = E_{31} - E_{13}, \quad e_3 = E_{32} - E_{23}, \\ e_4 &= -i(E_{23} + E_{32}), \quad e_5 = i(E_{13} + E_{31}), \quad e_6 = -i(E_{12} + E_{21}), \\ e_7 &= i(E_{22} - E_{11}), \quad e_8 = \frac{i}{\sqrt{3}}(2E_{33} - E_{11} - E_{22}). \end{aligned}$$

In the following subsections, we shall show that, from the cohomogeneity one $\mathrm{SU}(3)$ point of view, Wolf's quaternion-kähler manifolds arise by combining three basic models that correspond to tubular neighbourhoods

$G \times_H V$ of the relevant singular orbits G/H . These basic building blocks are summarised in Table 3.1, where Σ^2 is the irreducible 3-dimensional representation of $SU(2)$, $K = \Lambda^{2,0}$ denotes the 1-dimensional representation of $U(2)$ corresponding to the determinant, and \mathbb{R}^3 and \mathbb{C}^2 are the standard representations of $SO(3)$ and $U(2)$, respectively.

H	$\mathfrak{g}/\mathfrak{h}$	V	G/H
$SU(2)$	$\mathbb{R} \oplus \mathbb{H}$	Σ^2	S^5
$SO(3)$	$\odot_0^2 \mathbb{R}^3$	\mathbb{R}^3	L
$U(2)$	$[[\Lambda^{1,0}K]]$	\mathbb{C}^2	$\mathbb{C}P(2)$

TABLE 3.1. The three Wolf space building blocks.

In Table 3.1, L is the symmetric space $SU(3)/SO(3)$ that parametrises special Lagrangian subspaces of $\mathbb{R}^6 \cong \mathbb{C}^3$.

3.1. Quaternionic projective plane. In order to give a cohomogeneity one description of the quaternionic projective plane, we start by fixing the embedding of $SU(3)$ in $Sp(3)$ given via

$$SU(3) \subset \left\{ \begin{pmatrix} X & & \\ & 0 & \\ & & (X^{-1})^T \end{pmatrix} : X \in GL(3, \mathbb{C}) \right\} \subset Sp(3, \mathbb{C}).$$

Correspondingly, we have the following description of $\mathfrak{su}(3)$ at the Lie algebra level:

$$\mathfrak{su}(3) = \langle A_1 - A_2, A_2 - A_3, P_1, P_2, P_3, P_4, P_5, P_6 \rangle.$$

Now choose an element

$$Z \in \mathfrak{p} \cap \mathfrak{su}(3)^\perp = \langle Q_1, Q_2, Q_4, Q_5 \rangle.$$

We shall fix $Z = Q_1$. In these terms, the adapted quaternion-kähler frame pulls back as follows.

Lemma 3.1. *Under the mapping (3.1), Wolf's coframe pulls back to the dual of $\mathfrak{su}(3) \oplus \mathbb{R}$ to give*

$$\begin{aligned} \tilde{e}^1(t) &= 4\sqrt{2} \cos(2t)e^6, \quad \tilde{e}^2(t) = -4\sqrt{2} \cos(2t)e^7, \\ \tilde{e}^3(t) &= 4\sqrt{2}dt, \quad \tilde{e}^4 = \frac{4\sqrt{6}}{3} \sin(2t)e^8, \\ \tilde{e}^5(t) &= 4 \cos(t)(e^2 + e^4), \quad \tilde{e}^6(t) = 4 \cos(t)(e^3 + e^5), \\ \tilde{e}^7(t) &= 4 \sin(t)(e^2 - e^4), \quad \tilde{e}^8(t) = 4 \sin(t)(e^3 - e^5). \end{aligned} \tag{3.3}$$

Proof. In order to write things consistently with the structure equations (3.2), we fix on $\mathfrak{su}(3) \subset \mathfrak{sp}(3)$ the basis

$$\begin{aligned} e_1 &= A_1 - A_2, \quad e_2 = \frac{1}{\sqrt{2}}(P_2 - P_3), \quad e_3 = \frac{1}{\sqrt{2}}(P_5 + P_6), \quad e_4 = \frac{1}{\sqrt{2}}(P_2 + P_3), \\ e_5 &= \frac{1}{\sqrt{2}}(P_5 - P_6), \quad e_6 = P_1, \quad e_7 = -P_4, \quad e_8 = \frac{1}{\sqrt{3}}(A_1 + A_2 - 2A_3). \end{aligned}$$

Computing the action of $\text{Ad}(\exp(-tQ_1))$ with respect to the bases

$$e_1, \dots, e_8, \quad P_1, P_4, Q_1, Q_4, P_2, P_5, Q_2, Q_5$$

of $\mathfrak{su}(3)$ and \mathfrak{p} , respectively, we find that the map (3.1) is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cos(2t) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\cos(2t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3}\sqrt{3}\sin(2t) \\ 0 & \frac{1}{2}\cos(t)\sqrt{2} & 0 & \frac{1}{2}\cos(t)\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\cos(t)\sqrt{2} & 0 & \frac{1}{2}\cos(t)\sqrt{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2}\sin(t)\sqrt{2} & 0 & -\frac{1}{2}\sin(t)\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\sin(t)\sqrt{2} & 0 & -\frac{1}{2}\sin(t)\sqrt{2} & 0 & 0 & 0 \end{pmatrix}.$$

Considering its transpose, and recalling that $\frac{\partial}{\partial t}$ maps to Q_1 , we see that the adapted coframe f^1, \dots, f^8 pulls back to $\tilde{e}^1(t), \dots, \tilde{e}^8(t)$ to give the stated result. \square

Our computation confirms, more directly, the following result that is indicated in [12].

Proposition 3.2. *The Wolf space $\mathbb{H}\mathbb{P}(2)$ can be viewed as a cohomogeneity one manifold obtained by gluing together the disc bundles over the singular orbits $\mathbb{C}\mathbb{P}(2) = \mathrm{SU}(3)/\mathrm{U}(2)$ and $S^5 = \mathrm{SU}(3)/\mathrm{SU}(2)$. Each principal orbit is a copy of the exceptional Aloff-Wallach space $N^{1,-1} \cong N^{1,0}$.*

Proof. The point is to identify the principal and singular stabilisers; the latter appear at $t = 0$ and $t = \pi/4$. Since the singular orbits have codimension strictly smaller than 6, both singular stabilisers are connected [16, Corollary 1.9]. In particular, it suffices to work at the Lie algebra level, since connected subgroups of $\mathrm{SU}(3)$ are in one-to-one correspondence with subalgebras of $\mathfrak{su}(3)$. It is worthwhile making this more explicit by identifying the Lie algebras of the principal and singular stabilisers.

Regarding the principal orbits, we observe that the coframe (3.3) for generic t annihilates the $\mathfrak{u}(1)$ spanned by e_1 . When $t = 0$, the coframe is the annihilator of a Lie algebra $\mathfrak{u}(2)$ spanned by the four elements $e_1, e_8, e_2 - e_4, e_3 - e_5$. Finally, at $t = \pi/4$ the subspace annihilated is the $\mathfrak{su}(2)$ spanned by e_1, e_6, e_7 . \square

For later reference, let us emphasise that the tangent space of the open set corresponding to the principal orbits at each point decomposes as the $\mathrm{U}(1)$ -representation

$$\mathbb{R}^8 \cong 2\mathbb{R} \oplus 2V_1 \oplus V_2, \quad (3.4)$$

where V_k is the irreducible 2-dimensional representation on which the principal $\mathrm{U}(1)$ acts via matrices of the form

$$\begin{pmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{pmatrix};$$

at the infinitesimal level, this follows directly from (3.2).

As a final remark, note that $\mathbb{H}\mathbb{P}(2)$ comes with a $\mathrm{U}(1)$ -action, generated by the diagonal $\mathrm{U}(1)$ in $\mathrm{U}(3) \subset \mathrm{Sp}(3)$, that commutes with the action of $\mathrm{SU}(3)$. Clearly, this circle action generates a Killing vector field. Explicitly, this action is generated by $X = \sqrt{3}e_8 + 3A_3$ and in our cohomogeneity one

framework, it reads

$$\begin{aligned} \iota_t(g) \mapsto \exp(sX)\iota_t(g) &= g \exp(sX)\gamma(t)K = g \exp(\sqrt{3}se_8) \exp(3sA_3)\gamma(t)K \\ &= g \exp(\sqrt{3}se_8)\gamma(t) \exp(3sA_3)K = \iota_t(g \exp(\sqrt{3}se_8)), \end{aligned}$$

where we have used the fact that A_3 is an element of $\mathfrak{sp}(2) \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ that commutes with $Z = Q_1$. Therefore, the Killing vector field X can be identified with the left-invariant vector field $\sqrt{3}e_8$ on $SU(3)$.

3.2. Complex Grassmannian. In order to get an explicit description of the cohomogeneity one nature of $\text{Gr}_2(\mathbb{C}^4)$, we fix the copy of $SU(3)$ which comes from the usual embedding

$$SU(3) \cong \left\{ \begin{pmatrix} A & \\ & 1 \end{pmatrix} : A \in SU(3) \right\} \subset SU(4).$$

At the level of Lie algebras, this means that we are working with the copy $\mathfrak{su}(3) \subset \mathfrak{su}(4)$ spanned by

$$\begin{aligned} e_1 &= A_1 + A_2, e_2 = \frac{1}{\sqrt{2}}(C_1 + C_4), e_3 = \frac{1}{\sqrt{2}}(B_1 + B_4), e_4 = \frac{1}{\sqrt{2}}(C_1 - C_4), \\ e_5 &= \frac{1}{\sqrt{2}}(B_1 - B_4), e_6 = C_2, e_7 = -B_2, e_8 = \frac{1}{\sqrt{3}}(A_1 - A_2); \end{aligned}$$

this choice of a basis is consistent with (3.2). Proceeding as before, we pick an element

$$Z \in \mathfrak{p} \cap \mathfrak{su}(3)^\perp = \langle C_5, B_5, C_6, B_6 \rangle,$$

which, for concreteness, we shall fix to $Z = C_5$. Computations as in the proof of Lemma 3.1 then give:

Lemma 3.3. *Under the mapping (3.1), Wolf's coframe pulls back to the dual of $\mathfrak{su}(3) \oplus \mathbb{R}$ to give*

$$\begin{aligned} \tilde{e}^1(t) &= 2\sqrt{2} \cos(t)(e^2 + e^4), \tilde{e}^2(t) = 2\sqrt{2} \cos(t)(e^3 + e^5), \tilde{e}^3(t) = 4dt, \\ \tilde{e}^4(t) &= -\frac{4\sqrt{3}}{3} \sin(2t)e^8, \tilde{e}^5(t) = 4e^6, \tilde{e}^6(t) = -4e^7, \\ \tilde{e}^7(t) &= 2\sqrt{2} \sin(t)(-e^2 + e^4), \tilde{e}^8(t) = 2\sqrt{2} \sin(t)(e^3 - e^5). \end{aligned}$$

Using these observations, we have the following result.

Proposition 3.4. *The Wolf space $\text{Gr}_2(\mathbb{C}^4)$ can be viewed as a cohomogeneity one manifold obtained by gluing together the disc bundles over two copies of the singular orbit $\mathbb{C}P(2) = SU(3)/U(2)$. Each principal orbit is a copy of the exceptional Aloff-Wallach space $N^{1,0} \cong N^{1,-1}$. The form of the metric in the two bundles is the same up to the identification*

$$e_1 \mapsto -e_1, e_2 \mapsto e_4, e_3 \mapsto -e_5, e_4 \mapsto -e_2, e_5 \mapsto e_3, e_6 \mapsto e_6, e_7 \mapsto -e_7, e_8 \mapsto e_8.$$

Proof. The arguments follow those of Proposition 3.2. For convenience, let us write down the Lie algebras of the stabilisers. For the principal orbits, we have $\mathfrak{u}(1)$, corresponding to e_1 . At $t = 0$ we see that the algebra annihilated by the pulled back coframe is the copy of $\mathfrak{u}(2)$ spanned by $e_1, e_8, e_2 - e_4, e_3 - e_5$. Finally, at $t = \pi/2$ the Lie algebra of the singular stabiliser is spanned by the $\mathfrak{u}(2)$ determined by $e_1, e_8, e_2 + e_4, e_3 + e_5$.

The identification given is the inner automorphism of $\mathfrak{su}(3)$ obtained by conjugating with the matrix $\begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It preserves the principal $\mathfrak{u}(1)$ and the quaternion-kähler metric. The second statement follows. \square

As for $\mathbb{H}\mathbb{P}(2)$, note that the complex Grassmannian comes with a $U(1)$ -action, now generated by the diagonal $U(1)$ in $U(3) \subset SU(4)$, commuting with the $SU(3)$ -action. Again this circle action clearly generates a Killing vector field which in our cohomogeneity one framework can be identified with $e_8 \in \mathfrak{su}(3)$. In this case, $X = A_1 + 2A_2 + 3A_3$ is the sum of $-2\sqrt{3}e_8$ with $3A_1 + 3A_3$, which commutes with $Z = C_5$ and is contained in \mathfrak{K} .

3.3. The exceptional Wolf space. The quaternion-kähler structure on the space $G_2/SO(4)$ admits a cohomogeneity one description that comes from the embedding of $SU(3)$ in G_2 as the group generated by the long roots, cf. [18]. At the Lie algebra level, we have that $\mathfrak{su}(3)$ is generated by

$$A_1 + A_2, \frac{1}{\sqrt{2}}(W_2 + W_6), \frac{1}{\sqrt{2}}(Z_6 - Z_2), \frac{1}{\sqrt{2}}(W_2 - W_6), \\ -\frac{1}{\sqrt{2}}(Z_2 + Z_6), W_5, -Z_5, -\frac{1}{\sqrt{3}}(A_1 + 3A_2);$$

this choice of basis is consistent with (3.2). In order to study the orbits of the $SU(3)$ -action, we choose an element

$$Z \in \mathfrak{p} \cap \mathfrak{su}(3)^\perp = \langle W_3, W_4, Z_3, Z_4 \rangle.$$

Specifically, we fix $Z = W_3$. Then computations, completely similar to those in the proof of Lemma 3.1, give:

Lemma 3.5. *Under the mapping (3.1), Wolf's coframe pulls back to the dual of $\mathfrak{su}(3) \oplus \mathbb{R}$ to give*

$$\begin{aligned} \tilde{e}^1(t) &= \frac{\sqrt{2}}{2}(\cos(t)^3 - \sin(t)^3)e^2 + \frac{\sqrt{2}}{2}(\cos(t)^3 + \sin(t)^3)e^4, \\ \tilde{e}^2(t) &= \frac{\sqrt{2}}{2}(\cos(t)^3 - \sin(t)^3)e^3 - \frac{\sqrt{2}}{2}(\cos(t)^3 + \sin(t)^3)e^5, \\ \tilde{e}^3(t) &= -e^6, \tilde{e}^4(t) = e^7, \tilde{e}^5(t) = \sqrt{3}dt, \tilde{e}^6(t) = -\sin(2t)e^8, \\ \tilde{e}^7(t) &= -\sqrt{\frac{3}{8}}\sin(2t)(\sin(t) - \cos(t))e^2 - \sqrt{\frac{3}{8}}\sin(2t)(\sin(t) + \cos(t))e^4, \\ \tilde{e}^8(t) &= -\sqrt{\frac{3}{8}}\sin(2t)(\sin(t) - \cos(t))e^3 - \sqrt{\frac{3}{8}}\sin(2t)(\sin(t) + \cos(t))e^5. \end{aligned}$$

With the above observations, we have the following result that confirms statements from [12]:

Proposition 3.6. *The exceptional Wolf space $G_2/SO(4)$ can be viewed as a cohomogeneity one manifold obtained by gluing together disc bundles over the singular orbits $\mathbb{C}\mathbb{P}(2) = SU(3)/U(2)$ and $L = SU(3)/SO(3)$. Each principal orbit is an exceptional Aloff-Wallach space $N^{1,0} \cong N^{1,-1}$.*

Proof. Again the arguments are like those of Proposition 3.2, but for convenience we spell out the Lie algebras of the stabilisers. For the principal orbits, we have $\mathfrak{u}(1)$, corresponding to e_1 . At $t = 0$ we see that the algebra annihilated by the pulled back coframe is the copy $\mathfrak{u}(2)$ spanned by $e_1, e_8, e_2 - e_4, e_3 - e_5$. Finally, at $t = \pi/4$ the Lie algebra of the singular stabiliser is spanned by the copy $\mathfrak{so}(3)$ determined by e_1, e_2, e_3 . \square

In contrast with the quaternionic projective plane and the complex Grassmannian, $G_2/SO(4)$ clearly does not admit a (global) circle action from a commuting $U(1) \subset G_2$ ($SU(3) \subset G_2$ is a maximal connected subgroup). However, the open set corresponding to the principal orbits does come with a circle action corresponding to e_8 . Whilst X is not a Killing vector field in this case, it turns out to satisfy the generalised condition

$$d(\|X\|^2) \wedge \mathcal{L}_X \Omega = 0, \quad (3.5)$$

as follows by direct computation.

We conclude our cohomogeneity one description of the Wolf spaces with an observation that in a sense ties together all three cases.

Proposition 3.7. *The vector bundle $SU(3) \times_{U(2)} \mathbb{C}^2$ over $\mathbb{C}P(2)$ admits three distinct $SU(3)$ -invariant quaternion-kähler structures.*

Proof. First note that our analysis of the cohomogeneity one $SU(3)$ -actions shows that the spaces $\mathbb{H}P(2) \setminus S^5$, $Gr_2(\mathbb{C}^4) \setminus \mathbb{C}P(2)$ (one can choose either copy of $\mathbb{C}P(2)$) and $G_2/SO(4) \setminus L$ are all equivalent to the same vector bundle $SU(3) \times_{U(2)} \mathbb{C}^2$. The claim then follows since the three quaternion-kähler structures on $\mathbb{H}P(2)$, $Gr_2(\mathbb{C}^4)$ and $G_2/SO(4)$ induce different structures on $SU(3) \times_{U(2)} \mathbb{C}^2$, since these Wolf spaces have different holonomy groups and consequently different curvature. \square

4. NILPOTENT PERTURBATIONS

Let α be an element of $\Lambda^p(\mathbb{R}^n)^*$, and consider the (affine) perturbation by a fixed p -form δ , meaning

$$\beta(t) = \alpha + t\delta, \quad t \in \mathbb{R}.$$

Generally, it is hard to decide whether $\beta(t)$ and α lie in the same $GL(n, \mathbb{R})$ -orbit for all t . However, a useful sufficient criterion can be phrased as follows.

Proposition 4.1. *Let $A \in \mathfrak{gl}(n, \mathbb{R})$. If the associated derivation $\rho(A)$ satisfies $\rho(A)^2\alpha = 0$, then*

$$\beta(t) = \alpha + t\rho(A)\alpha$$

lies in the same $GL(n, \mathbb{R})$ -orbit as α for all $t \in \mathbb{R}$.

Proof. The proof is elementary. We expand $g(t) = \exp(tA)$ to find that

$$g(t)\alpha = \alpha + t\rho(A)\alpha,$$

since the higher order terms $\frac{t^k}{k!}\rho(A)^k\alpha$, $k \geq 2$, vanish by assumption. So $\beta(t)$ and α lie in the same $GL(n, \mathbb{R})$ -orbit, as claimed. \square

Motivated by Proposition 4.1, we would like to characterise the perturbations of α that are parametrised by solutions of

$$\rho(A)^2\alpha = 0. \quad (4.1)$$

Amongst these solutions we obviously have elements of the stabiliser \mathfrak{g} of α in $\mathfrak{gl}(n, \mathbb{R})$, but these give rise to trivial perturbations $\beta(t) \equiv \alpha$. In order to eliminate this indeterminacy we observe the following:

Proposition 4.2. *Let $\alpha \in \Lambda^p(\mathbb{R}^n)^*$. Then every solution A of (4.1) satisfies*

$$\rho(A)\alpha = \rho(N)\alpha,$$

where N is a nilpotent solution of (4.1).

Proof. Over the complex numbers we can put A , as an endomorphism of $(\mathbb{C}^n)^*$, into Jordan form. Correspondingly, we obtain a direct sum decomposition $(\mathbb{C}^n)^* = \bigoplus V_i$, where V_i is the generalised eigenspace relative to the eigenvalue λ_i . Denoting by I_{V_i} the matrix corresponding to the projection onto V_i , we have $A = N + \sum \lambda_i I_{V_i}$, where N is nilpotent and real.

In accordance with the above, we can also decompose $\Lambda^p(\mathbb{C}^n)^*$ as a direct sum

$$\bigoplus_{k_1+\dots+k_m=p} \bigoplus_{j_1<\dots<j_m} \Lambda^{k_1} V_{j_1} \otimes \dots \otimes \Lambda^{k_m} V_{j_m}, \quad (4.2)$$

where each summand is closed under $\rho(A)$. Moreover, direct computation shows that

$$\begin{aligned} & (\rho(A) - (\lambda_1 + \lambda_2)I)(\alpha_1 \wedge \alpha_2) \\ &= (\rho(A) - \lambda_1 I)\alpha_1 \wedge \alpha_2 + \alpha_1 \wedge (\rho(A) - \lambda_2 I)\alpha_2, \end{aligned}$$

giving that each summand in (4.2) is contained in the generalised eigenspace of $k_1\lambda_{j_1} + \dots + k_m\lambda_{j_m}$ relative to $\rho(A)$. It follows that $\rho(A)^2$ also preserves the decomposition (4.2), and its kernel is contained in

$$\bigoplus_{j_1<\dots<j_m} \bigoplus_{\substack{k_1+\dots+k_m=p \\ k_1\lambda_{j_1}+\dots+k_m\lambda_{j_m}=0}} \Lambda^{k_1} V_{j_1} \otimes \dots \otimes \Lambda^{k_m} V_{j_m}.$$

On this space, $\rho(A)$ and $\rho(N)$ act in the same way. Therefore, if A is a solution of Equation (4.1), then

$$\rho(N)^2\alpha = 0 \quad \text{and} \quad \rho(N)\alpha = \rho(A)\alpha. \quad \square$$

Motivated by Proposition 4.2, we shall restrict our attention to nilpotent solutions of (4.1) and refer to these as *nilpotent perturbations*.

Whilst Proposition 4.2 is valid in general, our interest is the case where α is the quaternionic form (1.1). If e_1^0, \dots, e_8^0 denotes a standard basis of \mathbb{R}^8 , and e_0^i its dual, it turns out that solutions to the perturbation problem are most conveniently expressed in terms of the orthonormal basis (E^1, \dots, E^8) of $(\mathbb{R}^8)^*$ equal to

$$(e_0^8, \frac{\sqrt{3}}{2}e_0^2 - \frac{1}{2}e_0^6, -\frac{\sqrt{3}}{2}e_0^1 - \frac{1}{2}e_0^5, -\frac{\sqrt{3}}{2}e_0^5 + \frac{1}{2}e_0^1, \frac{\sqrt{3}}{2}e_0^6 + \frac{1}{2}e_0^2, -e_0^4, e_0^3, e_0^7). \quad (4.3)$$

In these terms, (1.1) reads

$$\begin{aligned} \Omega = & -E^{1247} + \sqrt{3}E^{1248} - E^{1256} - E^{1346} + E^{1357} + \sqrt{3}E^{1358} \\ & + 2E^{1458} - E^{1678} - E^{2345} + 2E^{2367} - \sqrt{3}E^{2467} + E^{2468} - E^{2578} \\ & - E^{3478} - \sqrt{3}E^{3567} - E^{3568}. \end{aligned}$$

A significant observation, that may at first not be fully appreciated, is that the stabiliser of E^{123} in $\text{Sp}(2)\text{Sp}(1)$ is $\text{SO}(3)$. Computations give:

Lemma 4.3. *The stabiliser of $\langle E^1, E^2, E^3 \rangle$ in $\mathrm{Sp}(2) \mathrm{Sp}(1)$ is the copy of $\mathrm{SO}(3)$ whose Lie algebra is spanned by the elements*

$$\begin{pmatrix} 0 & \frac{\sqrt{3}}{2}j \\ \frac{\sqrt{3}}{2}j & -j \end{pmatrix} - \frac{1}{2}R_j, \quad \begin{pmatrix} 0 & \frac{\sqrt{3}}{2}k \\ \frac{\sqrt{3}}{2}k & k \end{pmatrix} - \frac{1}{2}R_k, \quad \begin{pmatrix} \frac{3}{2}i & 0 \\ 0 & -\frac{1}{2}i \end{pmatrix} - \frac{1}{2}R_i$$

of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$.

Above, R_i denotes right multiplication by i , and so forth.

Prompted by Lemma 4.3, we shall decompose $(\mathbb{R}^8)^*$ as the sum of two irreducible $\mathrm{SO}(3)$ -modules

$$S^2 = \langle E^1, E^2, E^3 \rangle, \quad S^4 = \langle E^4, \dots, E^8 \rangle.$$

Using the dual basis E_i , we then have that the 2-forms

$$\beta^1 := (E_2 \wedge E_3) \lrcorner \Omega, \quad \beta^2 := (E_3 \wedge E_1) \lrcorner \Omega, \quad \beta^3 := (E_1 \wedge E_2) \lrcorner \Omega \quad (4.4)$$

define an $\mathrm{SO}(3)$ -equivariant linear map $S^2 \rightarrow \Lambda^2 S^4$.

In these terms, the following result describes nilpotent perturbations in the quaternionic setting.

Theorem 4.4. *Up to the action of $\mathrm{Sp}(2) \mathrm{Sp}(1)$, nilpotent solutions of*

$$\rho(A)^2 \Omega = 0$$

are parametrised by linear maps $v: S^2 \rightarrow S^4$ such that

$$v^2 \wedge v^3 \wedge \beta^1 + v^3 \wedge v^1 \wedge \beta^2 + v^1 \wedge v^2 \wedge \beta^3 = 0.$$

Explicitly v corresponds to the endomorphism $A = \sum_{i=1}^3 v^i \otimes E_i$ of $(\mathbb{R}^8)^*$.

In terms of forms, the notation $v^i \otimes E_i$ above represents the endomorphism

$$\gamma \mapsto v^i \wedge (E_i \lrcorner \gamma).$$

Remark 4.5. *As a corollary of Theorem 4.4, nilpotent solutions of $\rho(A)^2 \Omega = 0$ actually satisfy $A^2 = 0$. Note that ρ is not an algebra homomorphism, so $A^2 = 0$ does not imply $\rho(A)^2 = 0$.*

The 3-dimensional subspace $\langle E_1, E_2, E_3 \rangle$ of \mathbb{R}^8 , i.e. the annihilator of S^4 , is uniquely determined up to the $\mathrm{Sp}(2) \mathrm{Sp}(1)$ -action. Subspaces of \mathbb{R}^8 in its $\mathrm{Sp}(2) \mathrm{Sp}(1)$ -orbit can be characterised by the angle between quaternionic lines. Indeed, consider the $\mathrm{Sp}(2) \mathrm{Sp}(1)$ -invariant function

$$Q: \mathbb{H}P^1 \times \mathbb{H}P^1 \rightarrow \mathbb{R}, \quad Q([v], [w]) = \max_{J \in \mathrm{Sp}(1)} \frac{\langle v, Jw \rangle^2}{|v|^2 |w|^2}.$$

In an affine chart we can express Q as

$$Q([1 : p], [1 : q]) = \frac{|1 + \bar{p}q|^2}{|1 + \bar{p}q|^2 + |q - p|^2}.$$

Lemma 4.6. *Let $v, w \in \mathbb{R}^8$ be two orthogonal non-zero vectors such that*

$$((v \wedge w) \lrcorner \Omega)^3 = 0.$$

Then $Q([v], [w]) = \frac{1}{4}$.

Proof. Using the action of the quaternionic unitary group, we can assume that $w = ae_8^0$ and $v = be_1^0 + ce_5^0$. Then a straightforward computation shows that $b = \pm\sqrt{3}c$, giving the asserted result. \square

Lemma 4.7. *Let w_1, \dots, w_k be orthonormal vectors in \mathbb{R}^8 such that*

$$((w_i \wedge w_j) \lrcorner \Omega)^3 = 0.$$

Then $k \leq 3$, and if $k = 3$, up to the action of $\mathrm{Sp}(2)\mathrm{Sp}(1)$, we may assume that $w_i = E_i$ for $i = 1, 2, 3$.

Proof. By Lemma 4.6, we have that

$$Q([w_i], [w_j]) = \frac{1}{4}, \quad i \neq j. \quad (4.5)$$

If $k > 2$, the points $[w_i]$ in $\mathbb{H}\mathbb{P}^1$ satisfy (4.5), and we can assume, up to $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -action, that

$$[w_1] = [1 : 0], [w_2] = [1 : p], [w_3] = [1 : q].$$

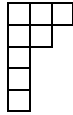
Then

$$|p|^2 = 3 = |q|^2, \quad |p - q|^2 = 3|1 + \bar{p}q|^2$$

which only has the solution $p = -q$. It is clearly not possible to add a fourth element $[w_4]$ so that (4.5) is satisfied.

Assuming then that $k = 3$, we can use the action of $\mathrm{Sp}(2)\mathrm{Sp}(1)$, as in the proof of Lemma 4.6, to obtain $w_1 = e_8^0$. This leaves us with an $\mathrm{SO}(4)$ symmetry that can be used to obtain $w_2 \in \langle e_1^0, e_2^0, e_3^0, e_4^0, e_6^0 \rangle$. The stabiliser in $\mathrm{SO}(4)$ of e_6^0 is $\mathrm{U}(2)$ and up to this $\mathrm{U}(2)$ -action, we can assume $w_2 = E_2$. The condition $p = -q$ together with orthogonality implies that w_3 is in the span of $\sqrt{3}e_1^0 + e_5^0$ and $\sqrt{3}e_3^0 + e_7^0$. The stabiliser of E_2 in $\mathrm{U}(2)$, isomorphic to $\mathrm{U}(1)$, acts non-trivially on this 2-dimensional space which allows us to set $w_3 = E_3$. \square

Proof of Theorem 4.4. Up to change of basis nilpotent matrices are classified over the reals by partitions with weight 8, giving 22 possibilities that can be encoded in terms of Young diagrams. For example, the diagram



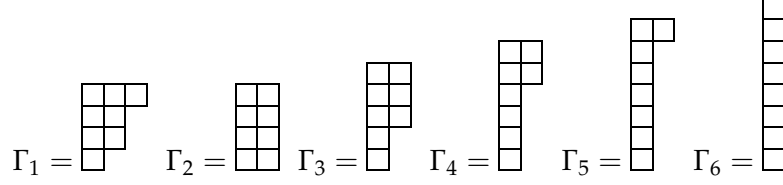
describes the endomorphisms of $(\mathbb{R}^8)^*$ with Jordan blocks of size 3, 2, 1, 1, 1, that with respect to some basis $\{w^1, \dots, w^8\}$ satisfy $w^3 \mapsto w^2$, $w^2 \mapsto w^1$, $w^5 \mapsto w^4$, with the other vectors mapped to zero. For each diagram Γ we can fix a representative endomorphism A_Γ and compute the space

$$K_\Gamma = \left\{ \alpha \in \Lambda^4(\mathbb{R}^8)^* : \rho(A_\Gamma)^2 \alpha = 0 \right\}.$$

The equation $\rho(A)^2 \Omega = 0$ has a solution with diagram Γ if $\rho(A_\Gamma)^2 \alpha = 0$ for some α in the orbit of Ω , and this requires that for each nonzero $v \wedge w$ in $\Lambda^2 \mathbb{R}^8$ the map

$$K_\Gamma \rightarrow \Lambda^4(\mathbb{R}^8)^*, \quad \alpha \mapsto ((v \wedge w) \lrcorner \alpha)^2$$

is not identically zero. Computations show that this rules out all cases except



The last diagram corresponds to $A = 0$ and Γ_5 corresponds to A being any rank one nilpotent matrix. In either case, the statement of the theorem holds.

Now let A be a solution of $\rho(A)^2\Omega = 0$. In terms of its diagram, let k be the number of rows of length greater than one, and reorder the associated basis in order that w^i corresponds to the rightmost box in the i th row for $1 \leq i \leq k$. In other words, the elements w_1, \dots, w_k of the dual basis span the annihilator $(\ker A + \operatorname{im} A)^\circ$. Then each $(w_i \wedge w_j) \lrcorner \Omega$ is degenerate, meaning that $((w_i \wedge w_j) \lrcorner \Omega)^3 = 0$. In the case of Γ_4 this holds because

$$2(Aw^1) \wedge (Aw^2) \wedge (w_1 \lrcorner w_2 \lrcorner \Omega) = \rho(A)^2\Omega = 0.$$

For $\Gamma_1, \Gamma_2, \Gamma_3$ the hypothesis implies degeneracy as it forces $w_i \lrcorner w_j \lrcorner w_\ell \lrcorner \Omega$ to be zero.

Without loss of generality, we can assume that the covectors w^i are orthogonal to $\operatorname{im} A$ and orthonormal. From Lemma 4.7 we conclude that $k \leq 3$, so that we can rule out Γ_2 , and assume $w_i = E_i$. In the case of Γ_3 and Γ_4 , it now suffices to write $A = \sum_{i=1}^3 v^i \otimes E_i$, so that, using (4.4), the vanishing of $\rho(A)^2\Omega$ becomes

$$v^{23} \wedge \beta^1 + v^{31} \wedge \beta^2 + v^{12} \wedge \beta^3 = 0,$$

as required.

Finally, in order to rule out the case Γ_1 , assume the associated basis has the form $E_1, E_2, E_3, v_1, \dots, v_5$, so that with obvious notation

$$A = v^1 \otimes E_1 + v^2 \otimes E_2 + v^3 \otimes E_3 + v^4 \otimes v_1.$$

Then

$$\begin{aligned} 0 &= \rho(A)^2\Omega \\ &= 2 \sum_{1 \leq i < j \leq 3} v^i \wedge v^j \wedge (E_i \lrcorner E_j \lrcorner \Omega) + v^4 \wedge (E_1 \lrcorner \Omega - 2 \sum_{i=1}^3 v^i \wedge (E_i \lrcorner v_1 \lrcorner \Omega)). \end{aligned} \tag{4.6}$$

Wedging with v^4 and using (4.4), we get

$$v^{124} \wedge \beta^3 + v^{314} \wedge \beta^2 + v^{234} \wedge \beta^1 = 0. \tag{4.7}$$

Projecting Equation (4.6) onto the space $S^2 \otimes \Lambda^3 S^4$, we find the condition

$$\begin{aligned} &v^4 \wedge \left[E^2 \wedge \beta^3 - E^3 \wedge \beta^2 + 2v^1 \wedge (v_1 \lrcorner (E^2 \wedge \beta^3 - E^3 \wedge \beta^2)) \right. \\ &\left. + 2v^2 \wedge (v_1 \lrcorner (E^3 \wedge \beta^1 - E^1 \wedge \beta^3)) + 2v^3 \wedge (v_1 \lrcorner (E^1 \wedge \beta^2 - E^2 \wedge \beta^1)) \right] = 0. \end{aligned}$$

This implies that

$$\begin{aligned} 0 &= v^4 \wedge [\beta^2 + 2v^1 \wedge (v_1 \lrcorner \beta^2) - 2v^2 \wedge (v_1 \lrcorner \beta^1)] \quad \text{and} \\ 0 &= v^4 \wedge [\beta^3 + 2v^1 \wedge (v_1 \lrcorner \beta^3) - 2v^3 \wedge (v_1 \lrcorner \beta^1)]. \end{aligned} \quad (4.8)$$

Therefore $v^{124} \wedge \beta^3 = v^{314} \wedge \beta^2 = 2v^{234} \wedge \beta^1$, and (4.7) implies each term is zero. Then Equation (4.8) gives $v^4 \wedge \beta^2, v^4 \wedge \beta^3 \in \langle v^{234} \rangle$, which is absurd. In conclusion, Γ_1 cannot occur, and the proof is complete. \square

4.1. U(1)-invariant perturbations. If we impose invariance, Theorem 4.4 can be simplified considerably. Indeed, consider the 8-dimensional representation of U(1) that models the tangent space to the open set formed of our principal orbits (cf. Equation (3.4)). The nilpotent perturbations compatible with this action have a simple description.

Proposition 4.8. *Let $\Omega \in \Lambda^4(\mathbb{R}^8)^*$ be a U(1)-invariant 4-form with stabiliser group $\text{Sp}(2)\text{Sp}(1)$. Then there is an orthonormal basis as in (4.3), such that (for the dual basis)*

$$E_1, E_8 \in 2\mathbb{R}, \quad 2V_1 = \langle E_2, E_3 \rangle \oplus \langle E_4, E_5 \rangle, \quad V_2 = \langle E_6, E_7 \rangle.$$

The space of U(1)-invariant nilpotent perturbations is generated by

$$E^8 \wedge (E_1 \lrcorner \Omega). \quad (4.9)$$

Proof. By construction the 4-form is fixed by U(1) so that this group lies inside $\text{Sp}(2)\text{Sp}(1)$. We can then find an adapted basis as in (4.3), such that $\text{U}(1) \subset \text{SO}(3) \subset \text{Sp}(2)\text{Sp}(1)$, where the middle subgroup $\text{SO}(3)$ preserves the splitting

$$\langle E^1, E^2, E^3 \rangle \oplus \langle E^4, \dots, E^8 \rangle = S^2 \oplus S^4;$$

this is because all subgroups U(1) in $\text{Sp}(2)\text{Sp}(1)$ are conjugate, so we can assume that U(1) is contained in some conjugate of $\text{SO}(3)$. Due to the way this U(1) acts on S^4 , we deduce that for the dual basis one has

$$\langle E_1, E_2, E_3 \rangle = \mathbb{R} \oplus V_1, \quad \langle E_4, \dots, E_8 \rangle = \mathbb{R} \oplus V_1 \oplus V_2.$$

Now, by making a change of basis if necessary, we can assume that

$$E_1 \in \mathbb{R}, \quad V_1 = \langle E_2, E_3 \rangle,$$

corresponding to U(1) stabilising E^1 in $\text{SO}(3)$, whose precise form can be recovered from Lemma 4.3. Computing its action on S^4 , we find

$$E_8 \in \mathbb{R}, \quad V_1 = \langle E_4, E_5 \rangle, \quad V_2 = \langle E_6, E_7 \rangle.$$

Since on nilpotent matrices the map $A \mapsto \rho(A)\Omega$ is injective, the latter is invariant if and only if A is invariant. This means that the space of invariant nilpotent perturbations is given by

$$v^1 \in \langle E^8 \rangle, \quad v^2, v^3 \in \langle E^4, E^5 \rangle,$$

where

$$\beta^1 \wedge v^{23} + \beta^2 \wedge v^{31} + \beta^3 \wedge v^{12} = 0.$$

It follows that $v^2 \wedge v^3 = 0$, and by invariance this means that $v^2 = 0 = v^3$. In conclusion, the space of U(1)-invariant perturbations is generated by (4.9), as required. \square

5. NEW CLOSED $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -STRUCTURES

We are now ready to produce explicit examples of closed $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structures. Since the corresponding exterior differential system is effectively underdetermined, it is not surprising that, at least locally, it is possible to obtain such examples by deforming the quaternion-kähler metric on a Wolf space M . In fact, it follows from results of the first author [7] that if one considers the induced structure on a real analytic hypersurface $N \subset M$ (in the language of [8], an $\mathrm{SO}(4)$ -structure with a closed 4-form β), then one can extend it to obtain a closed $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structure in a neighbourhood of N .

It is not difficult to see that there is more flexibility than that arising from local diffeomorphisms. In our cohomogeneity one setting, this indeterminacy can be seen by parametrising invariant forms in the $\mathrm{GL}(8, \mathbb{R})$ -orbit of the quaternion-kähler 4-form. These depend on 11 functions, because relatively to (3.4), the centralizer of $\mathrm{U}(1)$ in $\mathrm{GL}(8, \mathbb{R})$ has dimension 14 and intersects $\mathrm{Sp}(2)\mathrm{Sp}(1)$ in a 3-dimensional torus. Explicit computations show that closedness of the form corresponds to 7 equations, leaving 4 undetermined functions, whilst equivariant diffeomorphisms only depend on one function.

Whilst the discussion above emphasises local flexibility, the method of Section 4 proves to be a particularly useful approach to obtain examples that are both explicit and global. As it turns out, each Wolf space has a family of closed nilpotent perturbations determined by the vector field X , corresponding to e_8 . The perturbed metric happens to be genuinely different from the original only when this vector field is not Killing.

Lemma 5.1. *On each of the three Wolf spaces, $\mathrm{SU}(3)$ -invariant closed nilpotent perturbations of the quaternion-kähler structure Ω_{qK} have the form*

$$\tilde{\Omega} = \Omega_{qK} + dh \wedge (e_8 \lrcorner \Omega),$$

where h is any smooth $\mathrm{SU}(3)$ -invariant function.

Proof. An $\mathrm{SU}(3)$ -invariant perturbation is defined by an $\mathrm{SU}(3)$ -invariant section of $\mathrm{End}(T(\mathrm{G}/\mathrm{K}))$. On the complement of the singular orbits

$$\mathrm{SU}(3)/\mathrm{U}(1) \times (0, T),$$

Proposition 4.8 implies that the perturbation must be induced by a t -dependent nilpotent endomorphism of $\langle dt, e^8 \rangle$. Concretely, the perturbation must be of the form

$$\Omega_{qK} \pm (\lambda e^8 + \mu dt) \wedge \left((\mu e_8 - \lambda \frac{\partial}{\partial t}) \lrcorner \Omega \right).$$

Insisting that the perturbed 4-form is closed forces λ to vanish, since for all three quaternion-kähler metrics the restriction of $e^8 \wedge (\frac{\partial}{\partial t} \lrcorner \Omega)$ to principal orbits is not closed. On the other hand perturbations of the form $f(t)dt \otimes e_8$ preserve closedness, because $\mathcal{L}_{e_8} \Omega \wedge dt$ is zero in each case.

Having resolved the problem on the complement of singular orbits, we need to address the conditions that ensure that our solution will extend. Recall that there are three basic models to consider, summarised by Table 3.1. Let us first consider the vector bundle $\mathrm{SU}(3) \times_{\mathrm{U}(2)} \mathbb{C}^2$, where \mathbb{C}^2

is the standard representation of $U(2)$, corresponding to a tubular neighbourhood of each singular orbit $\mathbb{C}P(2)$. Away from the zero section, e_8 defines an invariant vector field. Since $e_8 \in \mathfrak{u}(2)$, this vector field is vertical, i.e. it is a $U(2)$ -invariant vector field on \mathbb{C}^2 . In appropriate real coordinates (x, y, z, w) , we can write

$$e_8 = \frac{2\sqrt{3}}{3} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z} \right), \quad t dt = x dx + y dy + z dz + w dw,$$

where the factor in front is due to the period of this $U(2)$ -invariant vector field.

This shows that $f(t)dt \otimes e_8$ extends smoothly if and only if $t \mapsto f(t)/t$ is smooth and even or, equivalently, $t \mapsto f(t)$ is smooth and odd. At tubular neighbourhoods of the other possible singular orbits, corresponding to the vector bundles $SU(3) \times_{SU(2)} \Sigma^2$ and $SU(3) \times_{SO(3)} \mathbb{R}^3$, respectively, e_8 defines an invariant direction in the isotropy representation. This means that $f(t)dt \otimes e_8$ is smooth if so is $f(t)dt$. Consequently, $t \mapsto f(t)$ must again be a smooth odd function. In summary, f extends to a smooth function on \mathbb{R} that satisfies

$$f(t) = -f(-t), \quad f(T-t) = -f(T+t).$$

Any primitive h of $f(t)dt$ then satisfies $h(t) = h(-t)$ and $h(T-t) = h(T+t)$ and therefore defines a global $SU(3)$ -invariant function on G/K , as required. \square

5.1. Perturbing with a Killing vector field. When X is Killing, we have rigidity in the sense that nilpotent perturbation just results in different, but $SU(3)$ -equivalent, ways of expressing Wolf's quaternion-kähler structure:

Proposition 5.2. *Applying $SU(3)$ -invariant nilpotent perturbations to the quaternion-kähler structure on $\mathbb{H}P(2)$ and $Gr_2(\mathbb{C}^4)$ leave the structures unchanged up to $SU(3)$ -equivariant isometry.*

Proof. It follows by Lemma 5.1 that each nilpotent perturbation can be associated with an $SU(3)$ -invariant function h and consequently an invariant vector field $-h(t)e_8$. Away from the singular orbits, its flow has the form

$$\phi^s: SU(3) \times (0, T) \rightarrow SU(3) \times (0, T), \quad (g, t) \mapsto (g \exp(-sh(t)e_8), t).$$

Then ϕ^1 is an equivariant diffeomorphism whose differential at (e, t) is given by

$$(v, 0) \mapsto (\text{Ad}(\exp(h(t)e_8))v, 0), \quad \frac{\partial}{\partial t} \mapsto -h'(t)e_8 + \frac{\partial}{\partial t}.$$

Since the adjoint action of e_8 preserves the quaternion-kähler metric, we obtain the same metric up to the isometry that corresponds to replacing e^8 by $e^8 + h'(t)dt$, as required. \square

5.2. Perturbing the exceptional Wolf space. As discussed X is not Killing in the case of $G_2/SO(4)$, but does satisfy the condition (3.5). In a sense this is exactly what is needed to obtain non-trivial perturbation results.

Theorem 5.3. *The exceptional Wolf space $G_2/\mathrm{SO}(4)$ admits $\mathrm{SU}(3)$ -invariant non-Einstein positive harmonic $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structures. The 4-form determining each such structure belongs to the same cohomology class as the quaternion-kähler 4-form.*

Proof. By Lemma 5.1, any $\mathrm{SU}(3)$ -invariant function h defines a closed perturbation

$$\Omega + dh \wedge (e_8 \lrcorner \Omega).$$

In order to verify that we get non-Einstein examples, we compute the Ricci tensor which equals

$$\begin{pmatrix} 8 - \frac{1}{3} \tan(2t)^2 h'(t)^2 & 0 & 0 & 0 & 0 & -\frac{1}{6} \frac{\sqrt{3} h'(t)^2 (3 + \cos(4t))}{\cos(2t)^2} - \frac{1}{3} \tan(2t) h''(t) - 4h'(t) & 0 & 0 & 0 & 0 \\ 0 & 8 - \frac{1}{3} \tan(2t)^2 h'(t)^2 & 0 & 0 & 0 & -\frac{1}{3} \tan(2t) h''(t) - 4h'(t) & \frac{1}{6} \frac{\sqrt{3} h'(t)^2 (3 + \cos(4t))}{\cos(2t)^2} & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 - \frac{4}{3} \tan(2t)^2 h'(t)^2 & \frac{4}{3} \tan(2t)^2 \sqrt{3} h'(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3} \tan(2t)^2 \sqrt{3} h'(t) & 8 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} \frac{\sqrt{3} h'(t)^2 (3 + \cos(4t))}{\cos(2t)^2} - \frac{1}{3} \tan(2t) h''(t) - 4h'(t) & 0 & 0 & 0 & 0 & 8 + \frac{1}{3} \tan(2t)^2 h'(t)^2 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} \tan(2t) h''(t) - 4h'(t) & \frac{1}{6} \frac{\sqrt{3} h'(t)^2 (3 + \cos(4t))}{\cos(2t)^2} & 0 & 0 & 0 & 0 & 8 + \frac{1}{3} \tan(2t)^2 h'(t)^2 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, note that this tells us that particular the scalar curvature is

$$s = 64 - \frac{4}{3} \tan(2t)^2 h'(t)^2, \quad (5.1)$$

so that we can get $s > 0$, but generally non-constant, by choosing h suitably.

For the final statement, we notice that the cohomology class of a closed 4-form on $G_2/\mathrm{SO}(4)$ is determined by its restriction to the singular orbit $\mathbb{C}\mathbb{P}^2$, which is a quaternionic submanifold. Since explicit verification shows that the quaternion-kähler form and perturbed 4-form both restrict to the volume form of $\mathbb{C}\mathbb{P}(2)$, we conclude that they belong to the same cohomology class, as required. \square

Remark 5.4. *Different choices of the perturbing function h in Theorem 5.3 yield non-isometric metrics. Indeed, let ϕ be an isometry between two such metrics. As both metrics have isometry group $\mathrm{SU}(3)$, ϕ maps $\mathrm{SU}(3)$ -orbits to $\mathrm{SU}(3)$ -orbits. In addition, corresponding orbits must have the same volume; notice that regardless of h , the volume of a principal orbit $\mathrm{SU}(3)/\mathrm{U}(1) \times \{t\}$ is a constant multiple of $\sin(2t)^3 \cos(2t)^2$. Therefore, the perturbing functions must coincide up to a constant.*

Remark 5.5. *As $\mathrm{SU}(3)$ -invariant functions on $G_2/\mathrm{SO}(4)$ are in one-to-one correspondence with smooth even functions of period $\pi/2$ on the reals, it is easy to find explicit closed perturbations where h is real-analytic.*

It is clear from Equation (5.1) that on $G_2/\mathrm{SO}(4)$ we only have constant scalar curvature in the quaternion-kähler case. If we are willing to remove the singular orbit $\mathbb{C}\mathbb{P}(2)$, however, the conclusion changes:

Corollary 5.6. *The vector bundle $\mathrm{SU}(3) \times_{\mathrm{SO}(3)} \mathbb{R}^3$ admits non-Einstein harmonic $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structures with constant scalar curvature.*

Proof. It follows from the proof of Theorem 5.3 that by choosing $h(t) = c \log \sin(2t)$, for any $c \in \mathbb{R}$, we obtain an incomplete closed $\mathrm{Sp}(2)\mathrm{Sp}(1)$ -structure of constant scalar curvature defined on $G_2/\mathrm{SO}(4) \setminus \mathbb{C}\mathbb{P}(2)$. For $c \neq 0$, this structure is non-Einstein. \square

6. RELATIONS TO OTHER SPECIAL GEOMETRIES

In our list of symmetric spaces with a cohomogeneity one $SU(3)$ -action one is missing, namely the Lie group $SU(3)$ itself, realised as the coset space $SU(3)^2/\Delta SU(3)$. The relevant action is by consimilarity [17]

$$SU(3) \times SU(3) \rightarrow SU(3): (g, h) \mapsto gh\bar{g}^{-1} = ghg^T.$$

This action preserves the parallel $PSU(3)$ -structure, given by

$$\gamma = \frac{1}{6} \sum_{i=1}^8 e^i \wedge de^i \quad (6.1)$$

in terms of our usual basis e^1, \dots, e^8 of $\mathfrak{su}(3)^*$.

Computations, similar to those of Section 3, reveal that there are equivariant isomorphisms $\mathbb{H}P(2) \setminus \mathbb{C}P(2) \cong SU(3) \setminus L$, and $G_2/SO(4) \setminus \mathbb{C}P(2) \cong SU(3) \setminus S^5$. A priori, the latter identification would seem to suggest the possibility of using the techniques of Section 4 to find new harmonic $PSU(3)$ -structures, as studied by Hitchin [15]. In fact, as for the exceptional Wolf space, $SU(3)$ has a “hidden” $U(1)$ -action, which has a natural interpretation in terms of the fibres of the equivariant map

$$SU(3) \ni P \mapsto P\bar{P},$$

which intertwines action by consimilarity and conjugation.

Computations show:

Proposition 6.1. *There are no non-trivial $SU(3)$ -invariant harmonic nilpotent perturbations of the $PSU(3)$ -structure (6.1) on $SU(3)$.*

Our studies are also related to G_2 -holonomy metrics. The starting point is the quotient of the quaternionic projective plane by the circle action generated by the Killing vector field X . More specifically, one has the $SU(3)$ -equivariant map

$$\mathbb{H}P(2) \setminus \mathbb{C}P(2) \rightarrow S^7 \setminus \mathbb{C}P(2) \cong \Lambda_-^2 \mathbb{C}P(2)$$

that appeared in [4], see also [20]. It is well known that the negative spinor bundle over $\mathbb{C}P(2)$ admits a complete metric with holonomy G_2 , the so-called Bryant-Salamon metric [6]. By building on work of [3, 13], the authors have succeeded in identifying the 3-form determining this G_2 -structure in terms of X , the 4-form and other quaternionic data. A more complete study will appear in a forthcoming paper.

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